

Nil geodesic triangles and their interior angle sums ^{*}

Jenő Szirmai

Budapest University of Technology and
Economics Institute of Mathematics,
Department of Geometry
Budapest, P. O. Box: 91, H-1521
szirmai@math.bme.hu

November 18, 2016

Abstract

In this paper we study the interior angle sums of geodesic triangles in Nil geometry and prove that it can be larger, equal or less than π .

We use for the computations the projective model of Nil introduced by E. Molnár in [7].

1 Introduction

A geodesic triangle in Riemannian geometry and more generally in metric geometry a figure consisting of three different points together with the pairwise-connecting geodesic curves. The points are known as the vertices, while the geodesic curves are known as the sides of the triangle.

In the geometries of constant curvature E^3 , H^3 , S^3 the well-known sums of the interior angles of geodesic triangles characterize the space. It is related to

^{*}Mathematics Subject Classification 2010: 53A20, 53A35, 52C35, 53B20.

Key words and phrases: Thurston geometries, Nil geometry, geodesic triangles, interior angle sum

the Gauss-Bonnet theorem which states that the integral of the Gauss curvature on a compact 2-dimensional Riemannian manifold M is equal to $2\pi\chi(M)$ where $\chi(M)$ denotes the Euler characteristic of M . This theorem has a generalization to any compact even-dimensional Riemannian manifold (see e.g. [2], [4]).

However, in the other 3-dimensional homogeneous maximal Riemann spaces (Thurston geometries) there are few results concerning the angle sums of geodesic triangles. Therefore, it is interesting to study similar question in the other five Thurston geometries, $S^2 \times \mathbf{R}$, $H^2 \times \mathbf{R}$, $\widetilde{\text{Nil}}$, $\widetilde{\text{SL}_2\mathbf{R}}$, Sol .

In [3] we investigated the angle sum of translation and geodesic triangles in $\widetilde{\text{SL}_2\mathbf{R}}$ geometry and proved that the possible sum of the internal angles in a translation triangle must be greater or equal than π . However, in geodesic triangles this sum is less, greater or equal to π .

In this paper we consider the analogous problem in Nil geometry.

Remark 1.1 *We note here, that nowadays the Nil geometry is a widely investigated space concerning the interesting surfaces, tilings, geodesic and translation ball packings (see e.g. [1], [5], [10], [11], [12], [13], [15], [16] and the references given there).*

In [1] K. Brodaczewska showed, that sum of the interior angles of translation triangles of the Nil space is larger than π .

Now, we are interested in *geodesic triangles* in Nil space that is one of the eight Thurston geometries [14, 18]. In Section 2 we describe the projective model of Nil and we shall use its standard Riemannian metric obtained by pull back transform to the infinitesimal arc-length-square at the origin. We also describe the isometry group of Nil, give an overview about geodesic curves.

In Section 3 we study the Nil geodesic triangles and prove the interior angle sum of a geodesic triangle in Nil geometry can be larger, equal or less than π .

2 Basic notions of the Nil geometry

In this Section we summarize the significant notions and denotations of the Nil geometry (see [7], [15]).

The Nil geometry is a homogeneous 3-space derived from the famous real matrix group $L(\mathbf{R})$ discovered by W. Heisenberg. The Lie theory with the methode of the projective geometry makes possible to investigate and to describe this topic.

The left (row-column) multiplication of Heisenberg matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+x & c+xb+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{pmatrix} \quad (2.1)$$

defines "translations" $\mathbf{L}(\mathbf{R}) = \{(x, y, z) : x, y, z \in \mathbf{R}\}$ on the points of the space $\mathbf{Nil} = \{(a, b, c) : a, b, c \in \mathbf{R}\}$. These translations are not commutative in general. The matrices $\mathbf{K}(z) \triangleleft \mathbf{L}$ of the form

$$\mathbf{K}(z) \ni \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto (0, 0, z) \quad (2.2)$$

constitute the one parametric centre, i.e. each of its elements commutes with all elements of \mathbf{L} . The elements of \mathbf{K} are called *fibre translations*. Nil geometry of the Heisenberg group can be projectively (affinely) interpreted by the "right translations" on points as the matrix formula

$$(1; a, b, c) \rightarrow (1; a, b, c) \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} = (1; x+a, y+b, z+bx+c) \quad (2.3)$$

shows, according to (2.1). Here we consider \mathbf{L} as projective collineation group with right actions in homogeneous coordinates. We will use the Cartesian homogeneous coordinate simplex $E_0(\mathbf{e}_0), E_1^\infty(\mathbf{e}_1), E_2^\infty(\mathbf{e}_2), E_3^\infty(\mathbf{e}_3), (\{\mathbf{e}_i\} \subset \mathbf{V}^4$ with the unit point $E(\mathbf{e} = \mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$ which is distinguished by an origin E_0 and by the ideal points of coordinate axes, respectively. Moreover, $\mathbf{y} = c\mathbf{x}$ with $0 < c \in \mathbf{R}$ (or $c \in \mathbf{R} \setminus \{0\}$) defines a point $(\mathbf{x}) = (\mathbf{y})$ of the projective 3-sphere \mathcal{PS}^3 (or that of the projective space \mathcal{P}^3 where opposite rays (\mathbf{x}) and $(-\mathbf{x})$ are identified). The dual system $\{(\mathbf{e}^i)\}, (\{\mathbf{e}^i\} \subset \mathbf{V}_4)$ describes the simplex planes, especially the plane at infinity $(\mathbf{e}^0) = E_1^\infty E_2^\infty E_3^\infty$, and generally, $\mathbf{v} = \mathbf{u}_c^\perp$ defines a plane $(\mathbf{u}) = (\mathbf{v})$ of \mathcal{PS}^3 (or that of \mathcal{P}^3). Thus $0 = \mathbf{x}\mathbf{u} = \mathbf{y}\mathbf{v}$ defines the incidence of point $(\mathbf{x}) = (\mathbf{y})$ and plane $(\mathbf{u}) = (\mathbf{v})$. Thus \mathbf{Nil} can be visualized in the affine 3-space \mathbf{A}^3 (so in \mathbf{E}^3) as well.

In this context E. Molnár [7] has derived the well-known infinitesimal arc-length square invariant under translations \mathbf{L} at any point of \mathbf{Nil} as follows

$$\begin{aligned} (dx)^2 + (dy)^2 + (-xdy + dz)^2 = \\ (dx)^2 + (1+x^2)(dy)^2 - 2x(dy)(dz) + (dz)^2 =: (ds)^2 \end{aligned} \quad (2.4)$$

Hence we get the symmetric metric tensor field g on \mathbf{Nil} by components g_{ij} , furthermore its inverse:

$$g_{ij} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+x^2 & -x \\ 0 & -x & 1 \end{pmatrix}, \quad g^{ij} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & x & 1+x^2 \end{pmatrix} \quad (2.5)$$

with $\det(g_{ij}) = 1$.

The translation group \mathbf{L} defined by formula (2.3) can be extended to a larger group \mathbf{G} of collineations, preserving the fibering, that will be equivalent to the (orientation preserving) isometry group of \mathbf{Nil} . In [8] E. Molnár has shown that a rotation through angle ω about the z -axis at the origin, as isometry of \mathbf{Nil} , keeping invariant the Riemann metric everywhere, will be a quadratic mapping in x, y to z -image \bar{z} as follows:

$$\begin{aligned} \mathbf{r}(O, \omega) : (1; x, y, z) &\rightarrow (1; \bar{x}, \bar{y}, \bar{z}); \\ \bar{x} &= x \cos \omega - y \sin \omega, \quad \bar{y} = x \sin \omega + y \cos \omega, \\ \bar{z} &= z - \frac{1}{2}xy + \frac{1}{4}(x^2 - y^2) \sin 2\omega + \frac{1}{2}xy \cos 2\omega. \end{aligned} \quad (2.6)$$

This rotation formula, however, is conjugate by the quadratic mapping \mathcal{M}

$$\begin{aligned} x \rightarrow x' = x, \quad y \rightarrow y' = y, \quad z \rightarrow z' = z - \frac{1}{2}xy \quad \text{to} \\ (1; x', y', z') \rightarrow (1; x', y', z') \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \omega & \sin \omega & 0 \\ 0 & -\sin \omega & \cos \omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (1; x'', y'', z''), \quad (2.7) \\ \text{with } x'' \rightarrow \bar{x} = x'', \quad y'' \rightarrow \bar{y} = y'', \quad z'' \rightarrow \bar{z} = z'' + \frac{1}{2}x''y'', \end{aligned}$$

i.e. to the linear rotation formula. This quadratic conjugacy modifies the \mathbf{Nil} translations in (2.3), as well. We shall use the following important classification theorem.

Theorem 2.1 (E. Molnár [8]) *1. Any group of \mathbf{Nil} isometries, containing a 3-dimensional translation lattice, is conjugate by the quadratic mapping in (2.5) to an affine group of the affine (or Euclidean) space $\mathbf{A}^3 = \mathbf{E}^3$ whose projection onto the (x, y) plane is an isometry group of \mathbf{E}^2 . Such an affine group preserves a plane \rightarrow point polarity of signature $(0, 0, \pm 0, +)$.*

2. Of course, the involutive line reflection about the y axis

$$(1; x, y, z) \rightarrow (1; -x, y, -z),$$

preserving the Riemann metric, and its conjugates by the above isometries in 1 (those of the identity component) are also Nil-isometries. There does not exist orientation reversing Nil-isometry.

Remark 2.2 We obtain from the above described projective model a new model of Nil geometry derived by the quadratic mapping \mathcal{M} . This is the linearized model of Nil space (see [1]).

2.1 Geodesic curves

The geodesic curves of the Nil geometry are generally defined as having locally minimal arc length between their any two (near enough) points. The equation systems of the parametrized geodesic curves $g(x(t), y(t), z(t))$ in our model can be determined by the general theory of Riemann geometry. We can assume, that the starting point of a geodesic curve is the origin because we can transform a curve into an arbitrary starting point by translation (2.1);

$$\begin{aligned} x(0) = y(0) = z(0) = 0; \quad \dot{x}(0) = c \cos \alpha, \quad \dot{y}(0) = c \sin \alpha, \\ \dot{z}(0) = w; \quad -\pi \leq \alpha \leq \pi. \end{aligned}$$

The arc length parameter s is introduced by

$$s = \sqrt{c^2 + w^2} \cdot t, \text{ where } w = \sin \theta, \quad c = \cos \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

i.e. unit velocity can be assumed.

The equation systems of a helix-like geodesic curves $g(x(t), y(t), z(t))$ if $0 < |w| < 1$:

$$\begin{aligned} x(t) &= \frac{2c}{w} \sin \frac{wt}{2} \cos \left(\frac{wt}{2} + \alpha \right), \quad y(t) = \frac{2c}{w} \sin \frac{wt}{2} \sin \left(\frac{wt}{2} + \alpha \right), \\ z(t) &= wt \cdot \left\{ 1 + \frac{c^2}{2w^2} \left[\left(1 - \frac{\sin(2wt + 2\alpha) - \sin 2\alpha}{2wt} \right) + \right. \right. \\ &\quad \left. \left. + \left(1 - \frac{\sin(2wt)}{wt} \right) - \left(1 - \frac{\sin(wt + 2\alpha) - \sin 2\alpha}{2wt} \right) \right] \right\} = \\ &= wt \cdot \left\{ 1 + \frac{c^2}{2w^2} \left[\left(1 - \frac{\sin(wt)}{wt} \right) + \left(\frac{1 - \cos(2wt)}{wt} \right) \sin(wt + 2\alpha) \right] \right\}. \end{aligned} \tag{2.8}$$

In the cases $w = 0$ the geodesic curve is the following:

$$x(t) = c \cdot t \cos \alpha, \quad y(t) = c \cdot t \sin \alpha, \quad z(t) = \frac{1}{2} c^2 \cdot t^2 \cos \alpha \sin \alpha. \quad (2.9)$$

The cases $|w| = 1$ are trivial: $(x, y) = (0, 0)$, $z = w \cdot t$.

Definition 2.3 The distance $d(P_1, P_2)$ between the points P_1 and P_2 is defined by the arc length of geodesic curve from P_1 to P_2 .

3 Geodesic triangles

We consider 3 points A_1, A_2, A_3 in the projective model of Nil space (see Section 2). The *geodesic segments* a_k between the points A_i and A_j ($i < j$, $i, j, k \in \{1, 2, 3\}$, $k \neq i, j$) are called sides of the *geodesic triangle* with vertices A_1, A_2, A_3 . In Riemannian geometries the metric tensor (see (2.5)) is used to define the

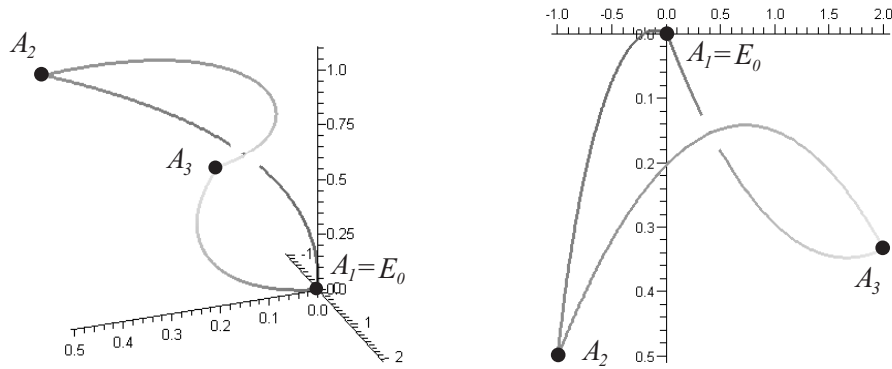


Figure 1: Geodesic triangle with vertices $A_1 = (1, 0, 0, 0)$, $A_2 = (1, 1/2, -1, 1)$, $A_3 = (1, 1/3, 2, 1)$ where its interior angle sum is $\approx 3.45294 > \pi$

angle θ between two geodesic curves. If their tangent vectors in their common point are u and v and g_{ij} are the components of the metric tensor then

$$\cos(\theta) = \frac{u^i g_{ij} v^j}{\sqrt{u^i g_{ij} u^j} \sqrt{v^i g_{ij} v^j}} \quad (3.1)$$

It is clear by the above definition of the angles and by the metric tensor (2.5), that the angles are the same as the Euclidean ones at the origin of the projective model of Nil geometry.

We note here that the angle of two intersecting geodesic curves depend on the orientation of the tangent vectors. We will consider the *internal angles* of the triangles that are denoted at the vertex A_i by ω_i ($i \in \{1, 2, 3\}$).

3.1 Fibre-like right angled triangles

A geodesic triangle is called fibre-like if one of its edges lies on a fibre line. In this section we study the right angled fibre-like triangles. We can assume without loss of generality that the vertices A_1, A_2, A_3 of a fibre-like right angled triangle (see Fig. 2) have the following coordinates:

$$A_1 = (1, 0, 0, 0), A_2 = (1, 0, 0, z^2), A_3 = (1, x^3, 0, z^3 = z^2) \quad (3.2)$$

The geodesic segment A_2A_3 lies on a to x axis parallel straight line, the geodesic

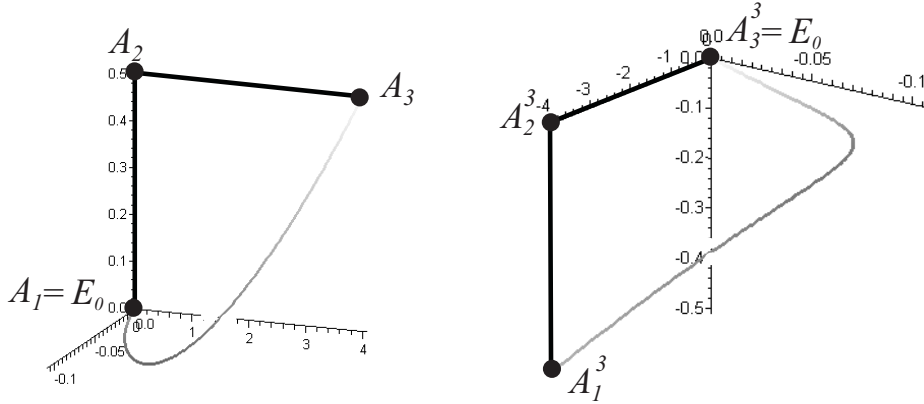


Figure 2: Fibre-like geodesic triangle $A_1A_2A_3$ and its translated image $A_1^3A_2^3A_3^3$ by translation T_{A_3} where $A_1 = (1, 0, 0, 0)$, $A_2 = (1, 0, 0, \frac{1}{2})$, $A_3 = (1, 4, 0, \frac{1}{2})$.

segment A_1A_2 lies on the z axis and their angle is $\omega_2 = \frac{\pi}{2}$ in the Nil space (this angle is in Euclidean sense also $\frac{\pi}{2}$) (see Fig. 2).

In order to determine the further internal angles of the fibre-like geodesic triangle $A_1A_2A_3$ we define translations \mathbf{T}_{A_i} , ($i \in \{2, 3\}$) as elements of the isometry group of Nil, that maps the origin E_0 onto A_i (see Fig. 2). E.g. the isometrie \mathbf{T}_{A_3} and its inverse (up to a positive determinant factor) can be given by:

$$\mathbf{T}_{A_3} = \begin{pmatrix} 1 & x^3 & 0 & z^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x^3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{T}_{A_3}^{-1} = \begin{pmatrix} 1 & -x^3 & 0 & -z^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -x^3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (3.3)$$

and the images $\mathbf{T}_{A_3}(A_i)$ of the vertices A_i ($i \in \{1, 2, 3\}$) are the following (see also Fig. 2):

$$\begin{aligned} \mathbf{T}_{A_3}^{-1}(A_1) &= A_1^3 = (1, -x^3, 0, -z^3), \quad \mathbf{T}_{A_3}^{-1}(A_2) = A_2^3 = (1, -x^3, 0, 0), \\ \mathbf{T}_{A_3}^{-1}(A_3) &= A_3^3 = E_0 = (1, 0, 0, 0). \end{aligned} \quad (3.4)$$

Our aim is to determine angle sum $\sum_{i=1}^3 (\omega_i)$ of the internal angles of the above

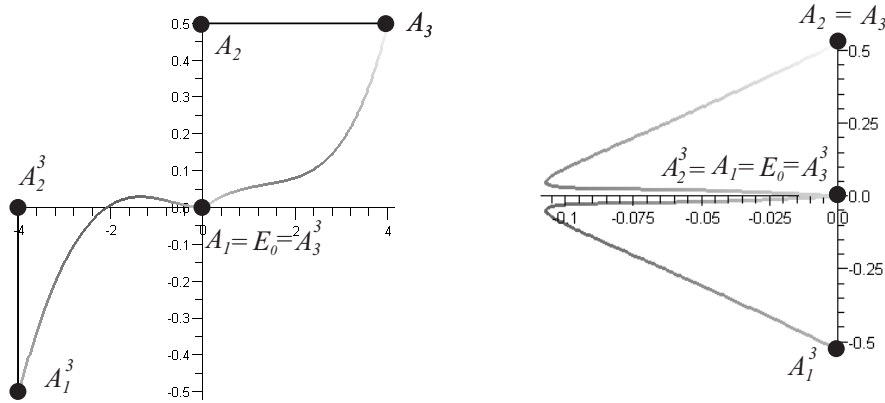


Figure 3: The points A_3 and A_1^3 are antipodal related to the origin E_0 ($A_1 = (1, 0, 0, 0)$, $A_2 = (1, 0, 0, \frac{1}{2})$, $A_3 = (1, 4, 0, \frac{1}{2})$.)

right angled fibre-like geodesic triangle $A_1A_2A_3$. We have seen that $\omega_2 = \frac{\pi}{2}$ and the angle of geodesic curves with common point at the origin E_0 is the same as the Euclidean one therefore it can be determined by usual Euclidean sense. Moreover, the translation \mathbf{T}_{A_3} is isometry in Nil geometry thus ω_3 is equal

to the angle $(g(A_3^3, A_1^3)g(A_3^3, A_2^3))\angle$ (see Fig. 2,3) where $g(A_3^3, A_1^3)$, $g(A_3^3, A_2^3)$ are oriented geodesic curves ($E_0 = A_2^2 = A_3^3$) and ω_1 is equal to the angle $(g(E_0, A_3)g(E_0, A_2))\angle$ (see Fig. 2,3) where $g(E_0, A_3)$, $g(E_0, A_2)$ are also oriented geodesic curves.

We denote the oriented unit tangent vectors of the oriented geodesic curves $g(E_0, A_i^j)$ with \mathbf{t}_i^j where $(i, j) = (1, 3), (2, 3), (3, 0), (2, 0)$ and $A_3^0 = A_3$, $A_2^0 = A_2$. The Euclidean coordinates of \mathbf{t}_i^j (see Section 2.1) are :

$$\mathbf{t}_i^j = (\cos(\theta_i^j) \cos(\alpha_i^j), \cos(\theta_i^j) \sin(\alpha_i^j), \sin(\theta_i^j)). \quad (3.5)$$

Lemma 3.1 *The sum of the interior angles of a fibre-like right angled geodesic triangle is greather or equal to π .*

Proof: It is clear, that $\mathbf{t}_2^0 = (0, 0, 1)$ and $\mathbf{t}_2^3 = (-1, 0, 0)$. Moreover, the points A_3 and A_1^3 are antipodal related to the origin E_0 therefore the equation $|\theta_3^0| = |\theta_1^3|$ holds (i.e. the angle between the vector \mathbf{t}_3^0 and $[x, y]$ plane are equal to the angle between the vector \mathbf{t}_1^3 and the $[x, y]$ plane). Moreover, we have seen, that $\omega_2 = \frac{\pi}{2}$. That means, that $\omega_1 = \frac{\pi}{2} - |\theta_3^0| = \frac{\pi}{2} - |\theta_1^3|$.

The vector \mathbf{t}_3^2 lies in the $[x, y]$ plane therefore the angle ω_3 is greather or equal than $|\theta_3^0| = |\theta_1^3|$. Finally we obtain, that

$$\sum_{i=1}^3 (\omega_i) = \frac{\pi}{2} + \frac{\pi}{2} - |\theta_3^0| + \omega_3 \geq \pi. \quad \square$$

We fix the $z^2 = z^3 \in \mathbf{R}$ coordinates of the vertices A_2 and A_3 and study the the internal angle sum $\sum_{i=1}^3 (\omega_i(x^3))$ of the right angled geodesic triangle $A_1 A_2 A_3$ if x^3 coordinate of vertex A_3 tends to zero or infinity. We obtain directly from the system of equation (2.8) of geodesic curves the following

Lemma 3.2 *If the coordinates $z^2 = z^3 \in \mathbf{R}$ are fixed then*

$$\lim_{x^3 \rightarrow 0} (\omega_1(x^3)) = 0, \quad \lim_{x^3 \rightarrow 0} (\omega_3(x^3)) = \frac{\pi}{2} \Rightarrow \lim_{x^3 \rightarrow 0} \left(\sum_{i=1}^3 (\omega_i(x^3)) \right) = \pi,$$

$$\lim_{x^3 \rightarrow \infty} (\omega_1(x^3)) = \frac{\pi}{2}, \quad \lim_{x^3 \rightarrow \infty} (\omega_3(x^3)) = 0 \Rightarrow \lim_{x^3 \rightarrow \infty} \left(\sum_{i=1}^3 (\omega_i(x^3)) \right) = \pi.$$

In the following table we summarize some numerical data of geodesic triangles for given parameters:

Table 1, $z^2 = z^3 = 1/2$					
x^3	$ \theta_3^0 = \theta_1^3 $	$d(A_1 A_3)$	ω_1	ω_3	$\sum_{i=1}^3 (\omega_i)$
$\rightarrow 0$	$\rightarrow \pi/2$	$1/2$	$\rightarrow 0$	$\rightarrow \pi/2$	$\rightarrow \pi$
$1/1000$	1.56878	0.50000	0.00202	1.56884	3.14166
$1/3$	0.97374	0.59901	0.59705	0.99435	3.16220
1	0.42883	1.10937	1.14197	0.48351	3.19627
4	0.05337	4.01337	1.51743	0.11957	3.20779
15	0.00169	15.00042	1.56911	0.01277	3.15268
100	0.00001	100.00000	1.57079	0.00030	3.14189
$\rightarrow \infty$	$\rightarrow 0$	$\rightarrow \infty$	$\rightarrow \pi/2$	$\rightarrow 0$	$\rightarrow \pi$

3.1.1 Hyperbolic-like right angled geodesic triangles

A geodesic triangle is hyperbolic-like if its vertices lie in the base plane (i.e. $[x, y]$ coordinate plane) of the model. In this section we analyse the internal angle sum of the right angled hyperbolic-like triangles. We can assume without loss of generality that the vertices A_1, A_2, A_3 of a hyperbolic-like right angled triangle (see Fig. 4) T_g have the following coordinates:

$$A_1 = (1, 0, 0, 0), A_2 = (1, 0, y^2, 0), A_3 = (1, x^3, y^2 = y^3, 0) \quad (3.6)$$

The geodesic segment $A_1 A_2$ lies on the y axis, the geodesic segment $A_2 A_3$ lies parallel to the x axis containing the point A_2 . It is clear that $\omega_2 = \frac{\pi}{2}$ in the Nil space (this angle is in Euclidean sense also $\frac{\pi}{2}$).

In order to determine the further internal angles of the fibre-like geodesic triangle $A_1 A_2 A_3$ similarly to the fibre-like case we define the translation \mathbf{T}_{A_3} , (see (2.6)) that maps the origin E_0 onto A_3 that can be given by:

$$\mathbf{T}_{A_3} = \begin{pmatrix} 1 & x^3 & y^2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x^3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{T}_{A_3}^{-1} = \begin{pmatrix} 1 & -x^3 & -y^2 & x^3 y^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -x^3 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.7)$$

We obtain that the images $\mathbf{T}_{A_3}^{-1}(A_i)$ of the vertex A_i ($i \in \{1, 2, 3\}$), are the following (see also Fig. 4):

$$\begin{aligned} \mathbf{T}_{A_3}^{-1}(A_1) = A_1^3 &= (1, -x^3, -y^2, x^3 y^2), \quad \mathbf{T}_{A_3}^{-1}(A_2) = A_2^3 = (1, -x^3, 0, 0), \\ \mathbf{T}_{A_3}^{-1}(A_3) &= A_3^3 = E_0 = (1, 0, 0, 0). \end{aligned} \quad (3.8)$$

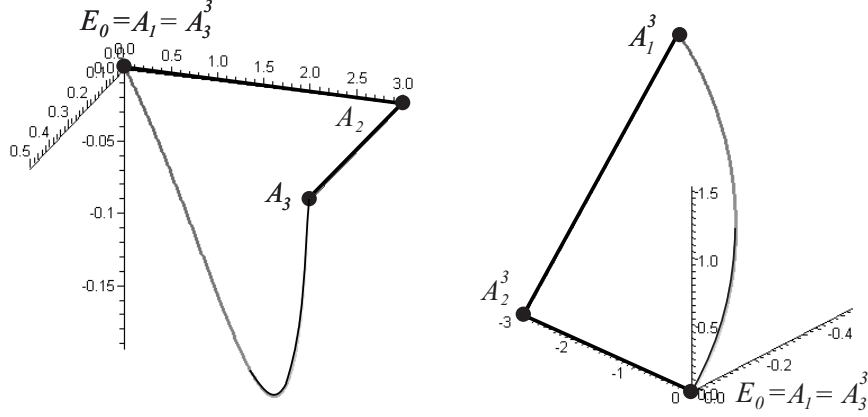


Figure 4: Hyperbolic-like geodesic triangle $A_1A_2A_3$ and its translated copy $A_1^3A_2^3A_3^3$.

We study similarly to the above fibre-like case the sum $\sum_{i=1}^3(\omega_i)$ of the interior angles of the above right angled hyperbolic-like geodesic triangle $A_1A_2A_3$.

It is clear, that the angle of geodesic curves with common point at the origin E_0 is the same as the Euclidean one therefore it can be determined by usual Euclidean sense. The translation \mathbf{T}_{A_3} preserve the measure of angles ω_i ($i \in \{2, 3\}$) therefore (see Fig. 4) $\omega_3 = (g(A_3^3A_1^3)g(A_3^3A_2^3))\angle(A_2^3 = A_3^3 = E_0 \text{ and } g(E_0A_i^j))$ ($(i, j) = (1, 3), (2, 3)$) are oriented geodesic curves).

Similarly to the fibre-like case the Euclidean coordinates of the oriented unit tangent vector \mathbf{t}_i^j of the oriented geodesic curves $g(E_0, A_i^j)$ ($(i, j) = (2, 3), (3, 2), (1, 2), (1, 3)$) is given by (3.5).

First we fix the $x^3 \in \mathbf{R}$ coordinate of the vertices A_3 and study the the internal angle sum $\sum_{i=1}^3(\omega_i(y^2 = y^3))$ of the right angled geodesic triangle $A_1A_2A_3$ if $y^2 = y^3$ coordinates of vertices A_2 and A_3 tend to zero or infinity. We obtain directly from the system of equation (2.8) of geodesic curves the following

Lemma 3.3 *If the coordinate $x^3 \in \mathbf{R}$ is fixed then*

$$\lim_{y^2=y^3 \rightarrow 0} (\omega_1(y^2)) = \frac{\pi}{2}, \quad \lim_{y^2=y^3 \rightarrow 0} (\omega_3(y^2)) = 0 \Rightarrow \lim_{y^2=y^3 \rightarrow 0} \left(\sum_{i=1}^3 (\omega_i(y^2)) \right) = \pi,$$

$$\lim_{y^2=y^3 \rightarrow \infty} (\omega_1(y^2)) = 0, \quad \lim_{y^2=y^3 \rightarrow \infty} (\omega_3(y^2)) = \frac{\pi}{2} \Rightarrow \lim_{y^2=y^3 \rightarrow \infty} \left(\sum_{i=1}^3 (\omega_i(y^2)) \right) = \pi.$$

Secondly we fix the $y^2 = y^3 \in \mathbf{R}$ coordinates of the vertices A_2 and A_3 and study the internal angle sum $\sum_{i=1}^3 (\omega_i(x^3))$ of the right angled geodesic triangle $A_1A_2A_3$ if x^3 coordinate of vertex A_3 tends to zero or infinity. We obtain directly from the system of equation (2.8) of geodesic curves the following

Lemma 3.4 *If the coordinates $y^2 = y^3 \in \mathbf{R}$ are fixed then*

$$\lim_{x^3 \rightarrow 0} (\omega_1(x^3)) = 0, \quad \lim_{x^3 \rightarrow 0} (\omega_3(x^3)) = \frac{\pi}{2} \Rightarrow \lim_{x^3 \rightarrow 0} \left(\sum_{i=1}^3 (\omega_i(x^3)) \right) = \pi,$$

$$\lim_{x^3 \rightarrow \infty} (\omega_1(x^3)) = \frac{\pi}{2}, \quad \lim_{x^3 \rightarrow \infty} (\omega_3(x^3)) = 0 \Rightarrow \lim_{x^3 \rightarrow \infty} \left(\sum_{i=1}^3 (\omega_i(x^3)) \right) = \pi.$$

We can determine the interior angle sum of arbitrary hyperbolic-like geodesic triangle similarly to the fibre-like case. In the following table we summarize some numerical data of geodesic triangles for given parameters:

Table 2, $x^3 = 1/2$					
$y^2 = y^3$	$ \theta_3^0 = \theta_1^3 $	$d(A_1A_3)$	ω_1	ω_3	$\sum_{i=1}^3 (\omega_i)$
$\rightarrow 0$	$\rightarrow \pi/2$	$1/2$	$\rightarrow \pi/2$	$\rightarrow 0$	$\rightarrow \pi$
1/100	0.00490	0.50011	1.54958	0.01940	3.13977
1/3	0.13378	0.60651	0.94882	0.56204	3.08166
3	0.13700	3.09310	0.14449	1.19813	2.91342
6	0.06132	6.06701	0.11959	1.30229	2.99268
20	0.00726	20.02442	0.04828	1.47308	3.09216
100	0.00030	100.00500	0.00999	1.55082	3.13160
$\rightarrow \infty$	$\rightarrow 0$	$\rightarrow \infty$	$\rightarrow 0$	$\rightarrow \pi/2$	$\rightarrow \pi$

Table 3, $y^2 = y^3 = 1/3$					
x^3	$ \theta_3^0 = \theta_1^3 $	$d(A_1A_3)$	ω_1	ω_3	$\sum_{i=1}^3 (\omega_i)$
$\rightarrow 0$	$\rightarrow \pi/2$	$1/3$	$\rightarrow 0$	$\rightarrow \pi/2$	$\rightarrow \pi$
1/100	0.00495	0.33349	0.02958	1.53998	3.14036
1/3	0.11518	0.47461	0.76510	0.76510	3.10100
3	0.09346	3.04190	1.31933	0.09854	2.98866
6	0.04131	6.02995	1.39094	0.08042	3.04215
20	0.00485	20.01086	1.50562	0.03222	3.10863
100	0.00020	100.00222	1.55748	0.00666	3.13493
$\rightarrow \infty$	$\rightarrow 0$	$\rightarrow \infty$	$\rightarrow \pi/2$	$\rightarrow 0$	$\rightarrow \pi$

Finally, we get the following Lemma

Lemma 3.5 *The interior angle sums of hyperbolic-like geodesic triangles can be less or equal to π .*

Conjecture 3.6 *The sum of the interior angles of any hyperbolic-like right angled hyperbolic-like right angled geodesic triangle is less or equal to π .*

3.1.2 Geodesic triangles with internal angle sum π

In the above sections we discuss the fibre- and hyperbolic-like geodesic triangles and proved that there are right angled geodesic triangles whose angle sum $\sum_{i=1}^3(\omega_i)$ is greater, less or equal to π , but $\sum_{i=1}^3(\omega_i) = \pi$ is realized if one of the vertices of a geodesic triangle $A_1A_2A_3$ tends to the infinity (see Tables 1,2,3). We prove the following

Lemma 3.7 *There is geodesic triangle $A_1A_2A_3$ with internal angle sum π where its vertices are proper (i.e. $A_i \in \text{Nil}$, ($i = 1, 2, 3$)).*

Proof: We consider a hyperbolic-like geodesic right angled triangle with vertices $A_1 = E_0 = (1, 0, 0, 0)$, $A_2^h = (1, 0, y_h^2, 0)$, $A_3^h = (1, x_h^3, y_h^2 = y_h^3, 0)$ and a fibre-like right angled geodesic triangle with vertices $A_1 = E_0 = (1, 0, 0, 0)$, $A_2^f = (1, 0, 0, z_f^2 = z_f^3)$, $A_3^f = (1, x_f^3, 0, z_f^2 = z_f^3)$ ($0 < x_h^3, x_f^3, y_h^2 = y_h^3, z_f^2 = z_f^3 < \infty$). We consider the straight line segment (in Euclidean sense) $A_3^fA_3^h \subset \text{Nil}$.

We consider a geodesic right angled triangle $A_1A_2A_3(t)$ where $A_3(t) \in A_3^fA_3^h$, ($t \in [0, 1]$). $A_3(t)$ is moving on the segment $A_3^hA_3^f$ and if $t = 0$ then $A_3(0) = A_3^h$, if $t = 1$ then $A_3(1) = A_3^f$.

Similarly to the above cases the internal angles of the geodesic triangle $A_1A_2A_3(t)$ are denoted by $\omega_i(t)$ ($i \in \{1, 2, 3\}$). The angle sum $\sum_{i=1}^3(\omega_i(0)) < \pi$ and $\sum_{i=1}^3(\omega_i(1)) > \pi$. Moreover the angles $\omega_i(t)$ change continuously if the parameter t run in the interval $[0, 1]$. Therefore there is a $t_E \in (0, 1)$ where $\sum_{i=1}^3(\omega_i(t_E)) = \pi$. \square

We obtain by the Lemmas of this Section the following

Theorem 3.8 *The sum of the internal angles of a geodesic triangle of Nil space can be greater, less or equal to π .*

References

- [1] Brodaczevska, K.: Elementargeometrie in \mathbf{Nil} . *Dissertation (Dr. rer. nat.) Fakultät Mathematik und Naturwissenschaften der Technischen Universität Dresden* (2014).
- [2] Chavel, I.: *Riemannian Geometry: A Modern Introduction*. Cambridge Studies in Advances Mathematics, (2006).
- [3] Csima, G, Szirmai, J.: Interior angle sum of translation and geodesic triangles in $\widetilde{\mathbf{SL}_2\mathbf{R}}$ space. *Submitted Manuscript* (2016).
- [4] Kobayashi, S., Nomizu, K.: *Fundation of differential geometry, I.* Interscience, Wiley, New York (1963).
- [5] Inoguchi, J.: Minimal translation surfaces in the Heisenberg group \mathbf{Nil}_3 . *Geom. Dedicata* **161/1**, 221–231 (2012).
- [6] Milnor, J.: Curvatures of left Invariant metrics on Lie groups. *Advances in Math.* **21**, 293–329 (1976)
- [7] Molnár, E.: The projective interpretation of the eight 3-dimensional homogeneous geometries. *Beitr. Algebra Geom.* **38**(2), 261–288 (1997)
- [8] Molnár, E.: On projective models of Thurston geometries, some relevant notes on \mathbf{Nil} orbifolds and manifolds. *Sib. Electron. Math. Izv.*, **7** (2010), 491–498, <http://mi.mathnet.ru/semr267>
- [9] Molnár, E., Szirmai, J.: Symmetries in the 8 homogeneous 3-geometries. *Symmetry Cult. Sci.* **21**(1-3), 87–117 (2010)
- [10] Molnár, E., Szirmai, J., Vesnin, A.: Projective metric realizations of cone-manifolds with singularities along 2-bridge knots and links. *J. Geom.*, **95**, 91–133 (2009)
- [11] Molnár, E., Szirmai, J.: On \mathbf{Nil} crystallography, *Symmetry Cult. Sci.*, **17/1-2** (2006), 55–74.
- [12] Pallagi, J., Schultz B., Szirmai, J.: Equidistant surfaces in \mathbf{Nil} space, *Stud. Univ. Zilina. Math .Ser.*, **25** (2011), 31–40.

- [13] Schultz, B., Szirmai, J.: On parallelohedra of Nil-space, *Pollack Periodica*, **7. Supplement 1** (2012): 129-136.
- [14] Scott, P.: The geometries of 3-manifolds. *Bull. London Math. Soc.* **15**, 401–487 (1983)
- [15] Szirmai, J.: The densest geodesic ball packing by a type of Nil lattices. *Beitr. Algebra Geom.* **48**(2), 383–398 (2007)
- [16] Szirmai, J.: Lattice-like translation ball packings in Nil space. *Publ. Math. Debrecen* **80**(3-4), 427–440 (2012)
- [17] Szirmai, J.: A candidate to the densest packing with equal balls in the Thurston geometries. *Beitr. Algebra Geom.* **55**(2) 441–452 (2014)
- [18] Thurston, W. P. (and Levy, S. editor): *Three-Dimensional Geometry and Topology*. Princeton University Press, Princeton, New Jersey, vol. **1** (1997)